Numerical evidence for stretched exponential relaxations in the Kardar-Parisi-Zhang equation

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We present results from extensive numerical integration of the Kardar-Parisi-Zhang (KPZ) equation in 1 +1 dimensions aimed to check the long-time behavior of the dynamical structure factor of that system. Over a number of decades in the size of the structure factor we confirm scaling and stretched exponential decay. We also give an analytic expression that yields a very good approximation to the numerical data. Our result clearly favors stretched exponential decay over recent results claiming to yield the exact time dependent structure factor of the 1+1-dimensional KPZ system. We suggest a possible solution to that contradiction.

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Many interesting dynamical phenomena in condensed matter physics are described in terms of nonlinear field equations driven by noise. A long list of examples includes turbulence, critical dynamics, the dynamics of interacting polymers, ballistic deposition (as well as other growth models), etc. The Kardar-Parisi-Zhang (KPZ) equation that describes a growing surface under ballistic deposition [1,2] is such a model. This equation formulated in terms of a height function $h(\vec{r},t)$ driven by external noise is given by

$$\frac{\partial h(\vec{r},t)}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\vec{r},t), \tag{1}$$

where ν is a diffusion constant, λ is the coupling constant (that controls the sticking rate of the deposited material), and $\eta(\vec{r},t)$ is a noise term driving the equation that models the randomness of the falling material. The noise is usually chosen to be Gaussian, with zero mean and second moment

$$\langle \eta(\vec{r},t)\,\eta(\vec{r}',t')\rangle = 2D_0\delta^d(\vec{r}-\vec{r}')\,\delta(t-t'), \tag{2}$$

where d is the substrate dimension and D_0 specifies the noise amplitude.

The KPZ equation has been suggested 17 years ago [1] as an extension of the linear Edawards-Wilkinson equation [3], so that a lot of research has been done on the statistical properties of the surfaces that this equation grows. An extensive review of this work can be found in Refs. [4–7]. It is well known that KPZ surfaces are self-affine, and are well described by two scaling exponents, namely, the roughness exponent α and the dynamic exponent z. It turns out that in the KPZ system these two exponents are not independent. Due to symmetry of Eq. (1) with respect to infinitesimal tilting [4], the famous scaling relation $\alpha+z=2$ is established. Furthermore, for the special case when $d=1$, the existence of a fluctuation-dissipation theorem gives the exact result $\alpha=1/2$ and $z=3/2$.

In recent years there has been a growing interest in the dynamical properties of the KPZ system. A question of great interest regards the long-time behavior of the dynamical structure factor $\Phi_q(t)=\langle h_q(0)h_{-q}(t)\rangle_S$, where $h_q(t)$ is the Fourier transform of the height function $h(\vec{r},t)$, and $\langle\cdots\rangle_S$ denotes steady-state averaging over the noise. Notice that by definition $\Phi_q(0)=\phi_q$, where ϕ_q is the static two-point function.

Using a self-consistent approach, Schwartz and Edwards were able to predict a stretched exponential decay for $\Phi_q(t)$ [8,9]. Regarding the KPZ system, they found the following long-time asymptotic behavior

$$\Phi_q(t) \sim c\phi_q(\gamma q t^{1/z})^{(d-1)/2} e^{-\gamma q t^{1/z}}, \tag{3}$$

where c and γ are dimensionless constants (not necessarily 1), d is the dimension, and z is the dynamic exponent. The same asymptotic behavior was also predicted analytically by Colaiori and Moore [10] using a mode-coupling approach. Later, Colaiori and Moore solved numerically the mode-coupling equations in one dimension [11] and confirmed the asymptotic analysis for the long-time behavior. Surprisingly, they also found that $\Phi_q(t)$ decays to zero in an oscillatory manner—a fact that was not revealed by the analytical tools.

It should be stressed, however, that the above describe only approximations to a solution of the KPZ equation. Even an exact solution of either the self-consistent expansion (SCE) equation or the mode-coupling equation would provide only an approximation for the real, time-dependant structure factor $\Phi_q(t)$ of the KPZ equation. Indeed, a more recent publication by Prähofer and Spohn suggests that the envelope of the one-dimensional structure factor decays exponentially rather than as a stretched exponential [12]. They claim an exact solution for the time-dependant structure factor of another model in the universality class of the KPZ system, namely, the polynuclear growth model. The actual solution for the structure factor follows a number of well-defined steps involving some direct though complicated numerical calculations. Those calculations are reported to be performed with extremely high precision that seems to ensure that the final solution for the structure factor is not affected by inaccuracies in the numerical procedure. Although

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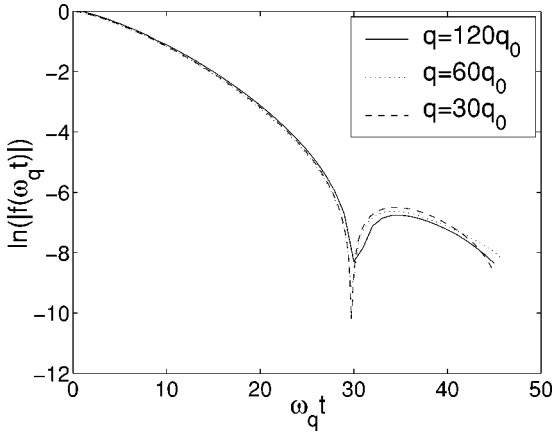

FIG. 1. A log plot of the scaling function $f(\omega_q t)$ for various small q's.

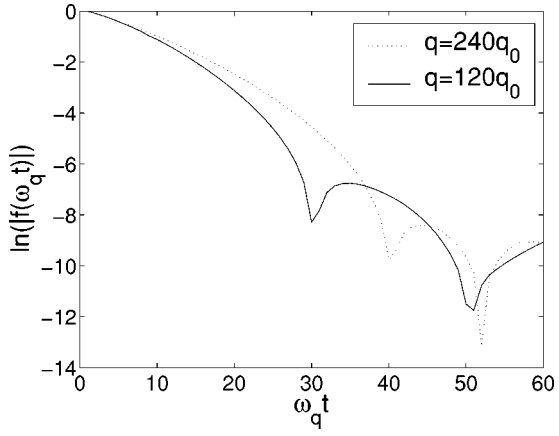

FIG. 2. A log plot of the scaling function $f(\omega_q t)$ for $q=120q_0$ and $q=120q_0$.

the solution they present is numerical, it is rather obvious that the envelope decay is exponential rather than stretched exponential.

This discrepancy motivated us to check the above results directly on the KPZ equation. In this work we find numerical support for the existence of stretched exponential relaxations in the KPZ system in $1+1$ dimensions in contradiction to Ref. [12]. We also get direct evidence for the oscillatory behavior of $\Phi_q(t)$. As a by product, we were able to verify the predicted short-time behavior of the dynamical structure factor (given in Ref. [13] for example), and the validity of the scaling hypothesis for small q's.

We discretized the KPZ equation (1) on a one-dimensional lattice, with lattice constant Δx, and time difference Δt,

$$h(x,t+\Delta t) = h(x,t) + \frac{\Delta t}{(\Delta x)^2}\sum_{i=1}^{d}\left\{\nu[h(x+\Delta x,t) - 2h(x,t) + h(x-\Delta x,t)] + \frac{\lambda}{8}[h(x+\Delta x,t) - h(x-\Delta x,t)]^2\right\} + \sigma(12\Delta t)^{12}\eta(t), \tag{4}$$

where $\sigma^2 \equiv 2D_0\Delta x$ and the random numbers $\eta(t)$ are uniformly distributed between $-1/2$ and $1/2$. In this work we used $L=1024$, $\Delta t=0.05$, $\Delta x=1$, $\nu=1$, $\lambda=4$, and $D_0=16$. After reaching steady state, at each time step we Fourier transformed the discrete height function, and obtained $h_q(t)$ for $q=q_0, 2q_0, \ldots$, where $q_0=2\pi/1024$. Then we calculated the two-point function $\langle h_q(0)h_{-q}(t)\rangle_S$ by averaging over all pairs with a time difference t, for $q=30q_0, 60q_0, 120q_0$, and $240q_0$.

First, the scaling hypothesis was numerically verified, i.e., in Fig. 1 we plot $f(\omega_q t)\equiv\Phi_q(t)/\phi_q$ as a function of $\omega_q t = Bq^z t$ with $z=3/2$ for $q=30q_0$, $q=60q_0$, and $q=120q_0$. The plot indicates good scaling.

However, we found that the scaling hypothesis breaks down when taking $q=240q_0$ (see Fig. 2) which means that the scaling, which is supposed to be correct for small q's, is correct only up to q's that are of order of $\sim 10\%$ of the largest q in the system.

Figure 1 indicates good scaling at least up to $120q_0$. Therefore, we invested most of the computational effort in this Fourier component, since Eq. (3) indicates that the larger q we take, the faster computational time evolution we get. For $q=120q_0$ we took 5×10^{10} integration time steps of Δt. Taking this component we found clear evidence for oscillatory behavior as shown in Fig. 3.

The error estimate in Fig. 3 was obtained using the imaginary part of $\langle h_q(0)h_{-q}(t)\rangle_S$ after averaging. Note that $\langle h_q(0)h_{-q}(t)\rangle_S$ should be real, due to averaging, while each contribution of the form $h_q(0)h_{-q}(t)$ is certainly not real. This means that the imaginary part, which should vanish eventually, is a sensible error estimate. An independent argument for the estimation of the error, which yields the same order of magnitude runs as follows: consider the case where there is no correlation at all and estimate the "apparent correlation" due to the finite sample. The total number of time steps is $N=5\times 10^{10}$. Therefore, this is also the number of pairs (h_q, h_{-q}) separated by a time $t=n\Delta t$ with $n \ll N$. The measured correlation is

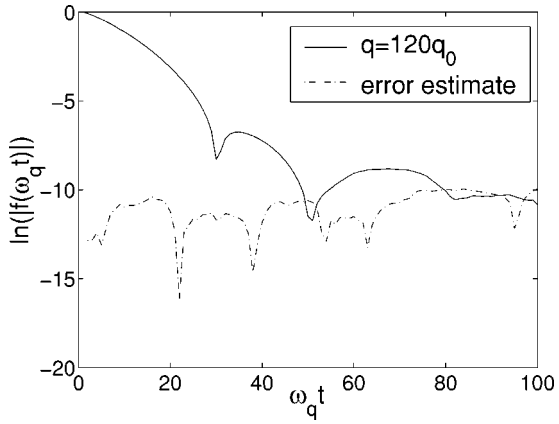

FIG. 3. A log plot of $f(\omega_q t)$ for $q=120q_0$ with an error estimate.

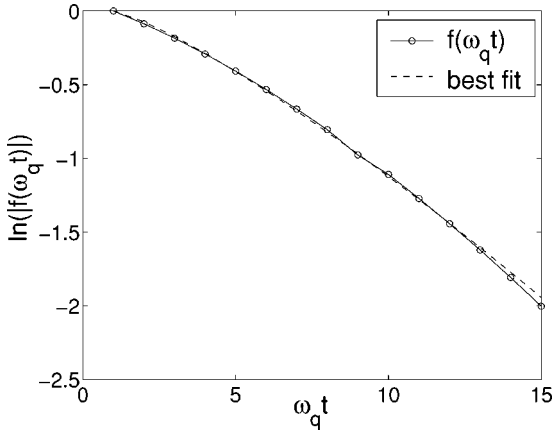

FIG. 4. A log plot of $f(\omega_q t)$ for small $\omega_q t$'s.

$$\langle h_q(0)h_{-q}(t)\rangle = \frac{1}{N}\sum_{i=1}^{N} h_q(i\Delta t)h_{-q}[(i+n)\Delta t]. \tag{5}$$

In the absence of correlation, the sum on the right-hand side of Eq. (5) is a sum of N random variables. Therefore, the size of the apparent correlation is of order $N^{-1/2}$, which is of the order of e^{-12}.

Using this result we extracted the short-time behavior of the scaling function. In Ref. [13] Colaiori and Moore predict $\Phi_q(t) \propto \phi_q[1-(\omega_q t)^{\Gamma/z}]$. Now, in one dimension $\Gamma=2$ (Γ is related to the roughness exponent α via $\Gamma=d+2\alpha$) and $z=3/2$, so that for a specific q we get $f(\omega_q t) \propto 1-(\omega_q t)^{4/3}$. This prediction was indeed verified by our data as shown in Fig. 4.

The error estimates presented in Fig. 3 suggest that the data may be useful in the range $0<\omega_q t<55$. Since we wanted to check a stretched exponential decay, we multiplied our numerical $f(\omega_q t)$ by $\exp[\gamma\omega_q t]^{2/3}$. We chose $\gamma=0.93$ so as to render the resulting function to be oscillating in that region. Motivated by the predictions of Ref. [13] and our previous verification of it we present the resulting function as a function of $(\omega_q t)^{4/3}$ in Fig. 5.

We were thus led to try the fit $e^{-\gamma(\omega_q t)^{2/3}}\{\cos[\beta(\omega_q t)^{4/3}] + D\sin[\beta(\omega_q t)^{4/3}]\}$ for the full scaling function $f(\omega_q t)$, as

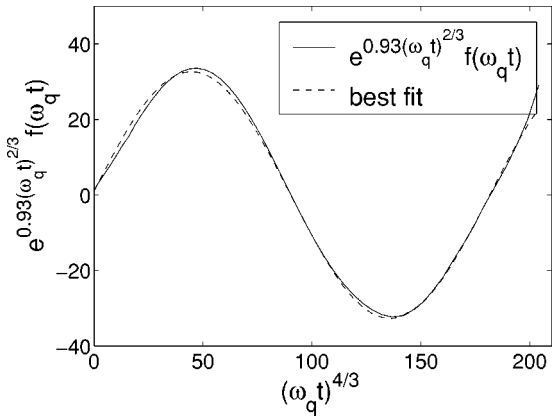

FIG. 5. A simple fit is $\{\cos[\beta(\omega_q t)^{4/3}]+D\sin[\beta(\omega_q t)^{4/3}]\}$ with $\beta=0.034$ and $D=32.7$.

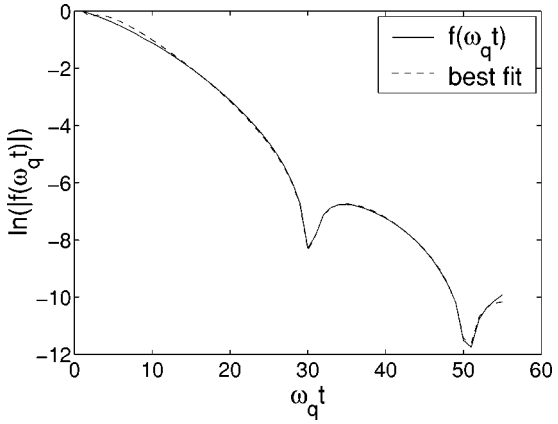

FIG. 6. A fit of $f(\omega_q t)$ using $e^{-\gamma(\omega_q t)^{2/3}}\{\cos[\beta(\omega_q t)^{4/3}] + D\sin[\beta(\omega_q t)^{4/3}]\}$ with $\gamma=0.93$, $\beta=0.034$, and $D=32.7$. Both curves practically join over most of the time range.

shown in Fig. 6. This has the right small $\omega_q t$ behavior as well as the envelope stretched exponential decay.

The fit is very good, especially when keeping in mind that this fit involves more than 6 orders of magnitude in the size of the scaling function Fig. 6. An attempt to replace the $2/3$ in the stretched exponential (in the ansatz) with an arbitrary exponent that will be determined by the fitting procedure yields a very close value ($\sim$0.65). Actually, we also tried to fit an exponential decay (rather than a stretched exponential one), according to the finding of Ref. [12], that gave a fit that was definitely worse (Fig. 7).

At this point we are facing an interesting contradiction. First, we have results following from four independent approaches. (1) Analytical asymptotic study of the self-consistent approximation. (2) Analytical asymptotic study of the mode-coupling approximation. (3) Numerical solution of the mode-coupling equations. (4) The present direct numerical integration of the KPZ equation. All these independent approaches yield the same stretched exponential decay. On the other hand, we have a publication [12] claiming to be exact that yields a result that is quite different from the results obtained from all the above mentioned methods. It is

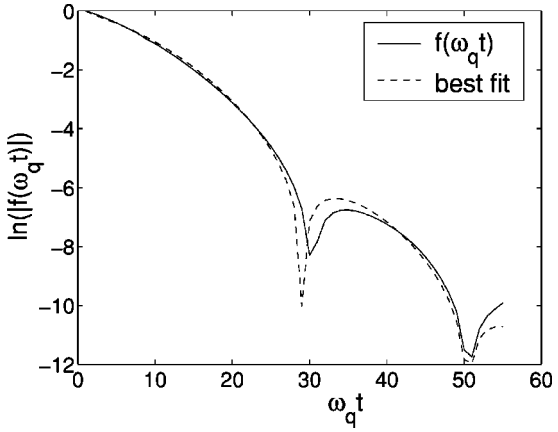

FIG. 7. A fit of $f(\omega_q t)$ using $e^{-\gamma(\omega_q t)}\{\cos[\beta(\omega_q t)^{4/3}] + D\sin[\beta(\omega_q t)^{4/3}]\}$. The best fit of that form is obtained for $\gamma=0.21$, $\beta=0.326$, and $D=2.6$.

possible that there is a flaw in the study presented in Ref. [12]. If there is, we have certainly not found any. We would like to suggest here another solution to the problem. Reference [12] does not consider directly the KPZ equation but rather the polynuclear growth model that was shown to be equivalent to the directed polymer problem with specific boundary conditions, and thus in the same universality class of the KPZ system. However, the point is that two models that are in the same universality class must have the same exponents but not necessarily the same scaling functions. In fact, it is possible to construct families of exactly solvable models where all the members of the family are characterized by the same exponents yet have radically different time-dependant structure factors. We will not go into this here but the interested reader could find the relevant ideas, although presented in a different context, in Ref. [14]. Our suggestion for solving the puzzle is thus that Prähofer and Spohn [12] obtain the correct decay for the polynuclear growth model, which is in itself a most impressive feat, but this is not the decay of the KPZ structure factor.

To summarize, using extensive numerical integration of the KPZ equation in $1+1$ dimensions, this work gives clear support for the scaling hypothesis (i.e., the fact that the scaling function $f(\omega_q t)$ is the same for any q, at least for small q's), verifies the short-time behavior given in Ref. [13] (i.e., $\Phi_q(t) \propto \phi_q[1-(\omega_q t)^{\Gamma/z}]$) and establishes the oscillatory decay of the dynamical structure factor to zero (as suggested in Ref. [11]). In addition, we show that the stretched exponential describes the decay of the structure factor over six orders of magnitude in its size. This implies that the KPZ problem is likely to be a respected member of the family of systems that exhibit slow relaxations—thus opening the door for mutual influence between the community of surface growth and that of slow dynamics.

The present work consumed quite a few months of CPU time on a Pentium 4 machine. In spite of that, we believe that the results presented here should motivate a much heavier numerical effort to deal with a region with larger $\omega_q t$'s for the one-dimensional system and hopefully with the two-dimensional system.

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